THE APPLICATION OF THE NODAL FORCE CONCEPT IN YIELD-LINE ANALYSIS. An illustrative example

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ABSTRACT

The capacity of a cantilever slab to carry a point-load P on the free edge or at some distance from the free edge is studied by means of the yield-line theory. The equilibrium method in conjunction with the use of the nodal force theory can be applied when the load P is situated exactly on the free edge but not when the load P is situated at some distance a from the free edge. This fact is not really obvious but becomes clear when the problem is studied with the work method. From the expression of the internal work it can be found that it is not possible to get a local minimum with respect to the relation between the yield-line pattern and the distance a of the load from the free edge. This explains why the equilibrium method is not applicable.

Key-words: Concrete slabs, Yield-line analysis, Nodal forces.

1. INTRODUCTION

By introducing the so-called nodal forces Johansen /1/, already in 1931, made it possible to use yield-line analysis of concrete slabs with the equilibrium method. In this method the equilibrium of the individual elements into which the slab is divided by the yield-lines are considered. Consequently it is an alternative solution to the virtual work method that can be used for the whole slab without knowing the equilibrium conditions for the individual elements. Because of the fact that the whole slab must be considered in the work equation the solution will as a rule be more complex than the solution with the equilibrium method.

The formulation of the nodal forces was rephrased by Jones /2/ 1962 and in many ways it became more clear and straight forward for the designer to use the equilibrium method. The general formulation according to Jones is given below:

$$Q_{12} = (m_{b3}-m_{b1})_3\cot\varphi_{13} - (m_{b3}-m_{b2})_3\cot\varphi_{23} + $$
$$+ (m_{t2}-m_{t1})_3 \qquad (1)$$

Thus Q_{12} is the nodal force between the yield-lines 1 and 2 in a point where three yield-lines 1, 2 and 3 meet according to FIG. 1.

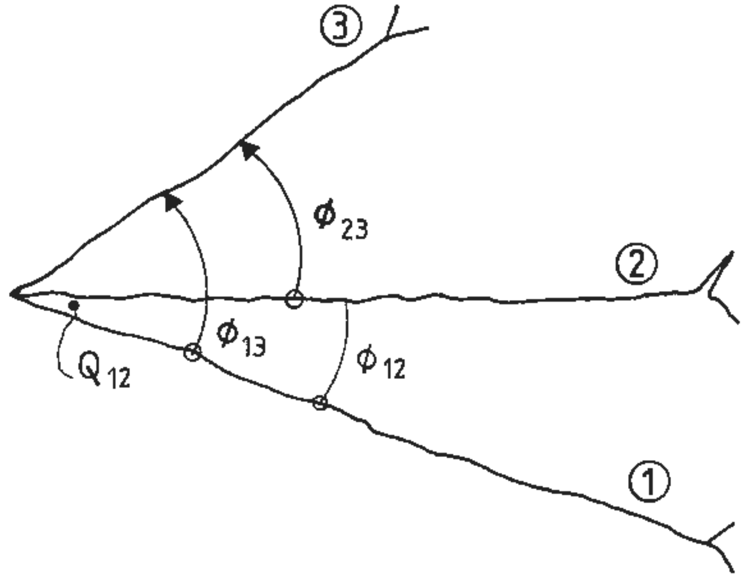

FIG. 1. The nodal force Q_{12} is related to the angles φ_{13} and φ_{23}.

between the yield lines ①, ② and ③ and to the bending- and twisting moments due to the reinforcement associated with the yield-lines.

The angles φ_{12} and φ_{13} are also shown in FIG. 1. The bending moments for the three yield-lines are determined by three different reinforcement meshes and m_{b13} is consequently the bending moment due to the reinforcement associated with yield-line 1 in the direction of yield-line 3. In the same way m_{t13} is the twisting moment associated with yield-line 1 in the direction of yield-line 3. All other symbols used are easily interpreted in the same way.

In the derivation of the nodal force formula (1) some restrictions are introduced. The nodal forces according to (1) correspond to a stationary maximum of the bending moments and when the yield-lines are attracted by a point-load or any geometrical irregularities the formula is not valid. It is not always easy to decide if the formula (1) can be used or not. One tricky case that the authors have studied is presented below.

2. PRESENTATION OF THE CASE

In his thesis /3/ from 1943 Johansen deals with the problem of a point-load P on the free edge of a long cantilever slab. From Johansens study it is obvious that the yield-line pattern according to FIG. 2 can be used. The reinforcement consists of one isotropic mesh in the upper face corresponding to the yield moment capacity m' and of one isotropic mesh in the lower face corresponding to the yield moment capacity m. The nodal forces can be formulated according to (1):

$$Q_1 = (m'+m)\cot\gamma$$

$$Q_2 = (m'+m)\cot\alpha \qquad (2)$$

$$Q_3 = m'\cot\beta$$

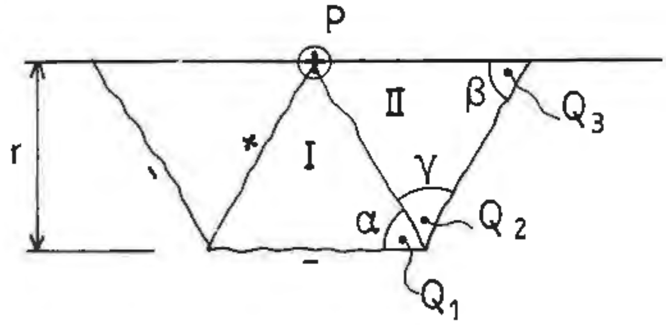

FIG. 2. The yield-line pattern for the case of a point-load on the free edge of a cantilever slab.

in agreement with Johansen's original treatment. The equilibrium for the slab element I claims that:

$$2(m'+m)\cdot r\cot\alpha = 2Q_1\cdot r$$

and therefore $\alpha = \gamma$. This implies moreover that $\beta = \pi-2\alpha$ and consequently:

$$\cot\beta = \frac{1-\cot^2\alpha}{2\cot\alpha} \qquad (3)$$

The equilibrium conditions for the slab element II can be formulated as (bending moments transverse to and parallel with the free edge):

$$(m'+m)r = Q_2\cdot r\cot\alpha + Q_3\cdot r(\cot\alpha+\cot\beta)$$

and

$$m'\cdot r\cot\beta - m\cdot r\cot\alpha = Q_2\cdot r$$

Both these conditions give as a result that:

$$\cot^2\alpha = \frac{m'}{3m'+4m} \qquad (4)$$

and finally that

$$P = 2(Q_1+Q_2+Q_3) = 2\sqrt{m'(3m'+4m)} \qquad (5)$$

This nodal force approach to the problem is thus quite straight forward and simple.

In practice the point-load P can be situated at some distance a from the free edge of the cantilever slab according to FIG. 3. It is for this case reasonable to assume an extended version of the yield-line pattern used for the case treated above. This assumed yield-line pattern is also sketched in FIG. 3. Even for this case it does not seem quite unreasonable to use the nodal force approach and this method is demonstrated as follows. The method is however not successful and in order to try to explain this the problem is also treated with the virtual work method.

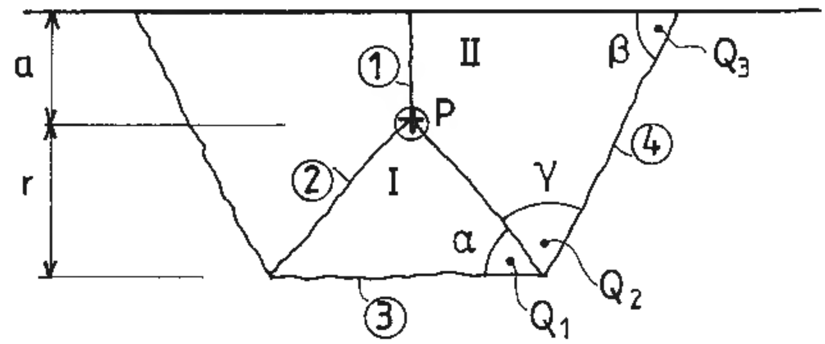

FIG. 3. The yield-line pattern for the case of a point-load at a distance a from the free edge of a cantilever slab.

3. NODAL FORCE APPROACH

If we use the nodal forces according to Eq. (2) the equilibrium for the slab element I requires that $\alpha = \gamma$ and we get the same relation between β and α as above formulated by Eq. (3). The equilibrium conditions for the slab element II (bending moments with respect to P transverse to and parallel with the free edge) can be formulated as:

$$\frac{a}{r} = \frac{(1-\cot^2\alpha)(4m\cot^2\alpha+3m'\cot^2\alpha-m')}{m'(1-\cot^2\alpha)^2-4(m+m')\cot^2\alpha} \qquad (6)$$

and

$$\frac{a}{r} = \frac{4m\cot^2\alpha+3m'\cot^2\alpha-m'}{2m'(1-\cot^2\alpha)} \qquad (7)$$

For $\frac{a}{r} = 0$ both the equations (6) and (7) give as a result that

$$\cot^2\alpha = \frac{m'}{3m'+4m}$$

in agreement with Eq. (4) above.

For $\frac{a}{r} \neq 0$ we get a necessary condition:

$$m'(1-\cot^2\alpha)^2 = -4(m+m')\cot^2\alpha \qquad (8)$$

which however is not possible to fulfil and we have to conclude that the nodal force approach is not applicable.

It is not really obvious that the nodal force approach should be applicable and in order to get some more insight we proceed and solve the problem with the virtual work method.

4. APPLICATION OF THE VIRTUAL WORK METHOD

The different yield-lines have been named ① , ② , ③ and ④ according to FIG. 3. By a virtual displacement = 1 of the load P the internal work in the yield-lines can be derived to be:

$$W_1 = 2m\cdot\frac{a}{r}\cdot\frac{1}{x+y}$$

$$W_2 = 2m\cdot\frac{1+x^2}{x+y}$$

$$W_3 = 2m'\cdot x$$

$$W_4 = 2m'(1+\frac{a}{r})\ \frac{1+y^2}{x+y}$$

In these expressions $x = \cot\alpha$ and $y = \cot\beta$. The maximum carrying capacity of the cantilever slab can than be obtained as the minimum of P where

$$P = W_1+W_2+W_3+W_4$$

From these expressions it is obvious that we can find local minima with respect to x and y but not with respect to r. Minimum of P corresponds to r being as large as possible and the yield-line ③ will coincide with the support line. In this situation the nodal force formulas are not applicable.

The solution of the problem can thus be found as the minimum with respect to x and y of:

$$P = \frac{1}{x+y}\cdot F(x,y) \qquad (9)$$

where

$$F(x,y) = 2m(1+x^2) + 2m'(1+y^2) + 2m'(x+y)x + \\ + 2m\cdot\frac{a}{r}+2m'(1+y^2)\cdot\frac{a}{r} \qquad (9a)$$

Minimum of P is obtained for:

$$\frac{\partial P}{\partial x} = \frac{1}{(x+y)^2}\left[(x+y)\cdot\frac{\partial F(x,y)}{\partial x} - F(x,y)\right] = 0$$

$$\frac{\partial P}{\partial y} = \frac{1}{(x+y)^2}\left[(x+y)\cdot\frac{\partial F(x,y)}{\partial y} - F(x,y)\right] = 0 \qquad (10)$$

Eq. (10) implies that

$$\frac{\partial F(x,y)}{\partial x} = \frac{\partial F(x,y)}{\partial y}$$

which gives the following relation between **x** and y:

$$\frac{x}{y} = \frac{m'}{m'+2m}\cdot\left(1+2\frac{a}{r}\right) \qquad (11)$$

Eq. (11) and one of the equations (10) can then be used to derive an expression of y^2:

$$y^2\cdot\frac{m'}{(m'+2m)^2}\left[4m+3m'+\left(4m+7m' + \frac{mm'}{m'+m}\right)\frac{a}{r} + \right.$$

$$\left. + \; 4m'\cdot\left(\frac{a}{r}\right)^2\right] = 1 + \frac{a}{r} \qquad (12)$$

For given values of m, m' and a/r the value of y is obtained from (12), x from (11) and finally the carrying capacity P from (9) and (9a).

The carrying capacity P has been calculated for m'/m = 1.0 and for m'/m = 2.0 and visualized in FIG. 4 as a function of a/r. For values of a/r > 1.0 other yield-line patterns are more relevant and this treated pattern should only be used for a/r ≤ 1.0.

The capacity P as a function of a/r and the bending capacities m and m' for a cantilever slab. Yield-line pattern according to FIG. 3.

5. SUMMARY

The capacity of a cantilever slab to carry a point-load P on the free edge or at some distance from the free edge is studied by means of the yield-line theory. The equilibrium method in conjunction with the use of the nodal force theory can be applied when the load P is situated exactly on the free edge but not when the load P is situated at some distance a from the free edge. This fact is not really obvious but becomes clear when the problem is studied with the work method. From the expression of the internal work it can be found that it is not possible to get a local minimum with respect to the relation between the yield-line pattern and the distance a of the load from the free edge. This explains why the equilibrium method is not applicable.

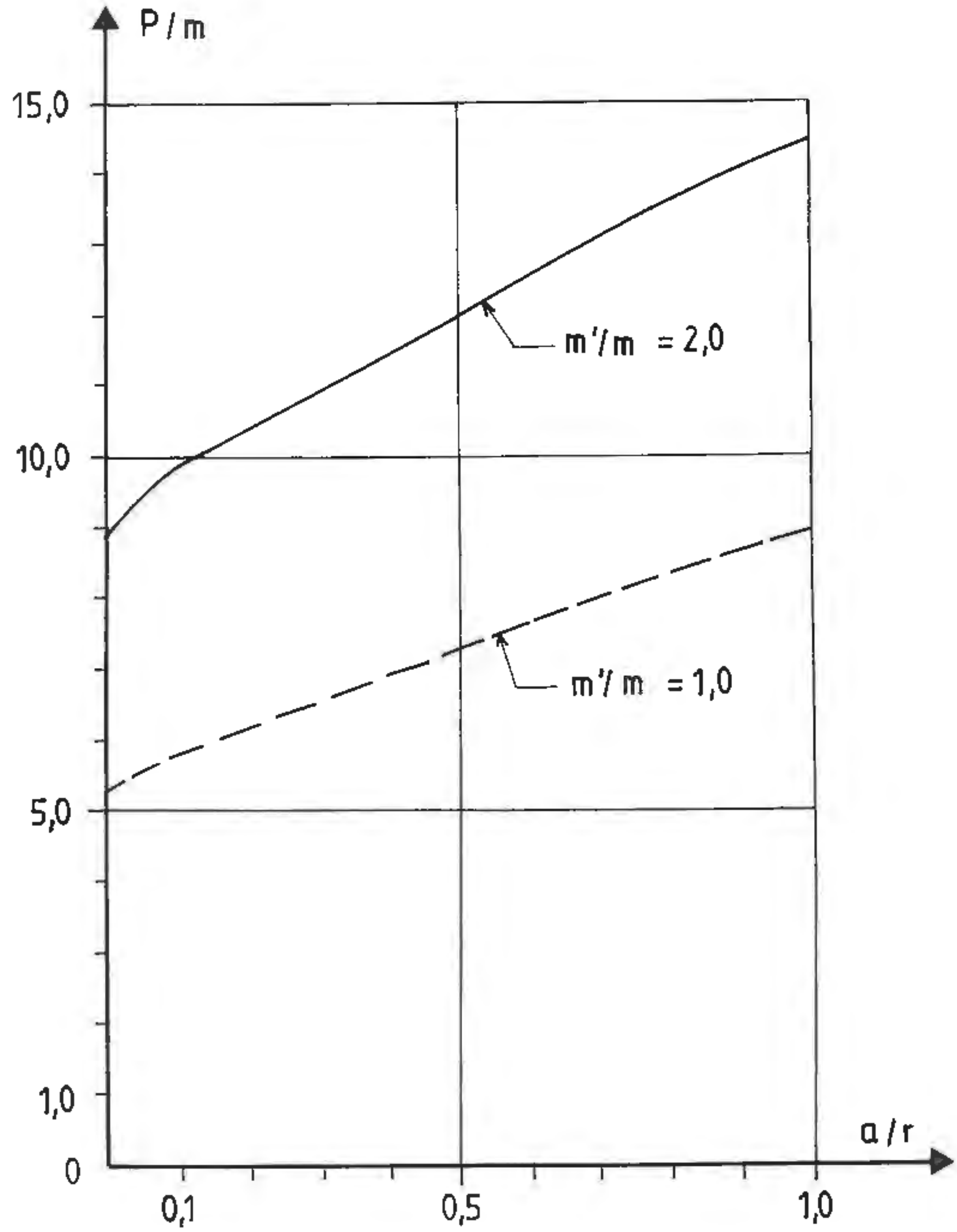

FIG. 4. Relation of P/m and parameter a/r

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