# METHOD TO ANALYSE AND DESIGN REINFORCED CONCRETE SHELL STRUCTURES

Jari Puttonen,
Imatran Voima Oy, Lic.Tech., Structural Analyst
Pekka Välikangas,
Imatran Voima Oy, M.Sc.Eng., Design Engineer

Synopsis

The objective of this study was to develop a computer program to analyse and design mechanically and thermally loaded reinforced concrete shell structures. The method considers the cracking of concrete and its effects on the thermal stresses. By the method the order of loading can also be studied. An important reason for developing this new design method is the prospect of combining designing of reinforced concrete structure and analysing based on the finite element method.

Keywords: design method, cracking, reinforced concrete

## 1. INTRODUCTION

Practical design of reinforced concrete structures bases on the national design codes which combine the demands of theoretical and practical engineering. The codes have their restrictions which sometimes cause difficulties to designers. A common problem is how to transform a three-dimensional state-of-stresses to the form implied by the codes. Especially the increasing use of the FEM (finite element method) in ordinary design has emphasized the problem. The other difficulty is cracking, and how this non-linear effect is taken into account. A present practice is to consider cracking only in the cross-sectional design. The reasons for this are simple; the glohal effects of cracking are difficult to estimate. Fortunately, these difficulties mostly affect the serviceability limit state. In the ultimate state the capacity may be examined by applying the theory of plasticity.

This study was carried out necessitated by the aforementioned difficulties encountered in the design of structures related to the nuclear energy industry. A common feature is that they are shell structures loaded by membrane and bending forces. However, the most prominent feature has been considerable thermal load and safety regulations which implied that the limit state of cracking was the most important design criterion. Thermal stresses are a direct function of the stiffness. Thus, the reduction of stiffness also decreases thermal forces. This means that the mechanism of cracking has an essential influence on the force distribution and on an optimal reinforcement. In general, a force distribution that is consistent with the theory of linear elasticity may double the optimal reinforcement in the case of thermal loads.

To solve the problem of cracking a computer program based on the non-linear theory can be used. Nowadays this approach is not economic, and furthermore the results of non-linear analyses might be too complicated to interpret in practical design. Gurfinkel /1/ has presented a method to consider cracking in the design of a cross-section loaded by a thermal gradient and mechanical forces. Varpasuo and Räty /2/ added to the method a way to analyse the constant change of temperature. The orientation of cracking in plates has been studied e.g. by Baumann /3/ and Brondum-Nielsen /4/. Gupta has completed these studies by introducing an exact mathematical expression for the principle of minimum resistance /5/, which is adopted also in the ACI-code /6/.

Relatively few attempts have been made to consider the influence of cracking on the force distribution in plate and shell structures. Seya et al. /7/ presented a method in which two force distributions are calculated. After the first analysis the membrane stiffness of the structure is evaluated assuming that the cracks occur in the directions of principal stresses. The second calculation is made with these new stiffnesses, and finally Gurfinkel's method is applied at the cross-sectional level. Gupta and Akbar /8/ used a similar idea but they calculated the directions of cracking more accurately. Both methods have been favourably compared with the results obtained by non-linear computer programs. The main restriction of these methods is that the reinforcement must be known beforehand. The membrane and bending condition have also been totally separated. Thus, the methods are not design procedures.

The basic idea in this study was to combine the analysis and design so that all the most essential features of the reinforced concrete structure are included. The design method bases on the theories of Gurfinkel and Gupta and Akbar. The analysis is made by an ordinary FEM-program, namely IVOFEM /9/, and the structure is described using four node linear shell elements which can be reduced to triangular elements. An element consists of a bending and a membrane element. The bending element is a combination of well-known LCCT 9 elements and the membrane condition is described by constant triangular elements. The method described is an iterative process. First the force distribution is calculated and then the structure is designed against these forces, i.e. its reinforcement is determined. During the design the stiffness of the structure is updated according to the cracking, and the obtained, now non-isotropic, stiffnesses are used in the next FEM-analysis. The final result is achieved when the reinforcement required does no longer change compared with the previous step. In the method both the reinforcement and the concrete are supposed to be elastic. The only non-linear effect is cracking. The ultimate state is considered by determining the reinforcement required from the linear force distribution in the beginning of the calculation. Later this reinforcement is used as a minimum requirement.

As a result of this study a computer program, IVODIM, was made. The program gives the reinforcement of the structure and it is fully synchronized with the IVOFEM. Thus, the designer, without difficulty to interpret its results, can utilize all the benefits of the FEM. The program has been tested by several examples, part of which are also reviewed here.

## 2. GENERAL DESCRIPTION OF THE CURRENT METHOD

To be able to use the method, the designers must know the following data: geometry of the structure, material properties of reinforcement and concrete, and loads. Thermal loads are given as surface temperatures. Through the structure the distribution of temperature is assumed to be linear. The industrial manufacturing of reinforcement requires logically arranged reinforcement consists of standard components. It is therefore possible to determine by the method that certain areas have similar reinforcement in spite of the stress distribution. The method can be used to analyse structures which act according to the theory of a thin shell. Mathematically, the structure is described by finite elements the curvature of which is infinite. The plate, that means pure bending, is calculated by using zero membrane stiffness. The membrane and bending stiffnesses are uncoupled during the whole process.

The program IVODIM has two main purposes. Firstly, it calculates the required reinforcement for each element and, secondly, the stiffness matrices are re-evaluated. As an input to the program we have the bending and membrane forces obtained from the FEM analysis. Two orthogonal directions of reinforcing bars have been given by the users. The design loads are obtained by transforming the force resultants of the FEM analysis to the directions of the reinforcement. This is done according to the principle of minimum resistance. Thus, the influence of cross-terms $N_{xy}$ and $M_{xy}$ is considered logically. As a result we have a design normal force and bending moment for both directions of the reinforcement. In the next the reinforcement required in the limit state of cracking is determined. This is done by applying the code practice. However, the real stiffness of the cross-section is carefully considered in this phase.

The other purpose of the IVODIM program is to form new stiffness matrices which include the influence of the cracking and the new reinforcement. The force distributions are according to the same FEM-analysis used in determining the reinforcement in the previous phase. The directions of the cracking are calculated using the method presented in Reference /8/. The basic idea is to analyse two steel layers which relates to the lower and upper surface of an element. The force resultants are transformed to the membrane forces of the steel layers. The thickness of a steel layer and the distance between the layers are those proposed by Baumann /3/

$$z = 0.85\ d \text{ and } t = z/3 \qquad (1)$$

where z, t and d are the distance of the layers, the thickness of a layer and the effective height of the cross-section, respectively.

On each steel layer the orientation of cracking is obtained as a numerical solution of an equation based on equilibrium and compatibility conditions /8/. One crack direction corresponding to the maximum tensile principal stress of both layers are calculated. By combining these cracks the principal direction of cracking is achieved. The difference of the directions of cracks varies from zero to 90 degrees between the layers. The former value relates to the situation of zero bending moments, and the latter one to the situation of zero membrane forces. A combined principal direction of cracking is calculated from the directions of the cracks. The method to combine the cracks depends on the difference between the crack directions. New stiffness matrices are formed in the system of coordinates, one axis of which coincides with the combined direction of cracking. The other axis differs 90 degrees from the direction of the first one. Both stiffnesses and forces are now rotated to this system of coordinates. Two cross-sections are analysed so that a coordinate axis is a normal of a cross-section. In this phase, Gurfinkel's method is applied. If the force distribution relates to uncracked concrete, we once calculate new thermal forces which correspond to the altered stiffnesses and update the direction of cracking according to these new thermal forces. As the final result of the IVODIM-program, elasticity matrices describing bending and membrane stiffnesses are obtained.

For uncracked concrete these matrices are identical, but for cracked concrete they are different. By using the elasticity matrices a new IVOFEM-analysis is msde. The iteration continues until the reinforcement after different FEM-analyses does no longer change.

The main restrictions of the method are that the structure is supposed to obey the theory of thin shells and the material is elastically linear, although cracking. The cracks occur when the tensile stress is greater than the tensile strength of concrete. The bending and membrane conditions are partly analysed separately. This may be justified only by the fact that the error made is small with small deformations. Some approximations are also made when considering cross-terms, like $N_{xy}$ and $M_{xy}$. The principle of minimum resistance is only one, even though good, way to solve the problem. Simplified methods are also used to derive the cross-stiffness terms needed.

3. DESIGN PROCEDURE

A key question in designing of a concrete plate or shell element is how the membrane forces $N_x$, $N_y$, $N_{xy}$ and bending moments $M_x$, $M_y$, $M_{xy}$ are considered. This subject has been dealt with in msny papers /3/, /10/. Relatively few researchers have tried to analyse membrane and bending forces simultaneously /4/, /11/. However, that should be the aim, since the principal directions of the membrane forces and the bending moments generslly do not coincide. Gupta's latest attempt to solve the problem by using the principle of minimum resistance is very promising /12/. In our program an older version of the principle of minimum resistance is used, in which the bending and membrane conditions are separated. The main reason was that these formulations have simple closed-form solutions.

As an example of the principle of minimum resistance the determination of required reinforcement against membrane forces will be given. Let us assume that an orthogonal reinforcement has the capacities $N_x^*$ and $N_y^*$. The problem is to minimize these capacities so that the element sustains the force distributions $N_x$, $N_y$ and $N_{xy}$. The next step is to express the external forces and the resisting forces in the critical direction $\theta$:

$$N_\theta^* = N_x^* \cos^2\theta + N_y^* \sin^2\theta \qquad (2)$$

$$N_\theta = N_x \cos^2\theta + N_y \sin^2\theta + 2N_{xy} \sin\theta\cos\theta \qquad (3)$$

According to the principle of minimum resistance the following equations can be written

$$\frac{d}{d\theta}\,(N_\theta^*/N_\theta) = 0; \quad N_\theta^* = N_\theta \qquad (4)$$

The equations (2) - (4) yield

$$N_x^* = N_x + N_{xy} \tan\theta \qquad (5)$$

$$N_y^* = N_y + N_{xy} \cot\theta \qquad (6)$$

Eliminating the angle from the equations (5) and (6) results in the fact that any reinforcement is allowable which satisfy the condition

$$(N_x^* - N_x)\,(N_y^* - N_y) \geq N_{xy}^2 \qquad (7)$$

The reinforcement is designed against the forces $N_x^*$ and $N_y^*$. To maintain the internal equilibrium the concrete is loaded by a compressive force, $N_c$ parallel to the crack. The value of this force is

$$N_c = - 2 N_{xy}/\sin 2\theta \qquad (8)$$

The equations (5) and (6) define design equations in any arbitrary direction of $\theta$. If the reinforcement is used to provide tensile resistance, the forces $N_x^*$ and $N_y^*$ are positive, and an optimum reinforcement is obtained by minimizing the sum of $N_x^* + N_y^*$. The solution of this problem is

$$\tan\theta = \cot\theta = N_{xy}/|N_{xy}| \qquad (9)$$

which gives the design forces

$$N_x^* = N_x + |N_{xy}| \qquad (10)$$

$$N_y^* = N_y + |N_{xy}| \qquad (11)$$

However, it should be checked that the optimized solution satisfies the boundary conditions. If the value of $N_x^*$ in the equation (10) is negative, the solution presented is not admissible. Now the solution will be obtained by setting $N_x^* = 0$ in the equation (5). Then we have

$$N_y^* = N_y - N_{xy}^2/N_x \qquad (12)$$

No tensile reinforcement is required if $N_y^*$ is negative in the equation (12). A similar approch is made if $N_y^*$ is negative in the equation (11). The derivation has been thorouhgly presented in Gupta's paper /5/ with the identical presentation for bending moments. In our program this kind of equations are used.

The characteristic width of cracking is calculated by applying a formula which bases on the CEB-FIB model code /13/

$$w_k = k_\theta \varepsilon_{sm} \, (3.5 \cdot c + k_w \phi/\phi_r) \qquad (13)$$

where $w_k$ is the characteristic crack width, $k_\theta$ is a coefficient to consider the oblique position between cracking and reinforcement, $\varepsilon_{sm}$ is medium steel strain between the cracks, c is the thickness of surface concrete, $k_w$ includes the influence of the bending of reinforcement and the deformation around the tensile reinforcement, $\phi$ is the medium diameter of steel bars and $\phi_r$ is a ratio of the area of tensile reinforcement to the assumed tensile zone of the cross-section. The value of $k_\theta$ is taken according to Leonhardt's results /14/. The values 75° and 45° of $\theta$ relate to the values 1.0 and 1.6 of $k_\theta$, respectively, where $\theta$ is the angel between the directions of reinforcement and cracking.

The medium steel strain is evaluated using the CEB-FIB formula /13/

$$\varepsilon_{sm} = \varepsilon_s \, (1-(25\, k_w)^{-1} \cdot (\sigma_{sr}/\sigma_s)^2) \gtrsim 0.4\varepsilon_s \qquad (14)$$

where $\varepsilon_s$ is the steel strain in the cracked cross-section. Two steel stresses are needed. The first one $\sigma_{sr}$ is the steel stress at the moment when cracks occur and the other $\sigma_s$ relates to the strain $\varepsilon_s$.

To determine the strains the equilibrium position of the cross-section should be solved. The neutral axis, which means a line where the strains are zero, is obtained from the equation

$$N(e'+a)+E_c\bar{I}\phi_T+E_c\bar{A}(a-\bar{x})\delta_T-(N+E_c\bar{A}\delta_T)I/S = 0 \qquad (15)$$

where a is the distance of the neutral axis from the compressed surface. I is the moment of inertia around the neutral axis and $\bar{I}$ is the moment of inertia around the axis of center of gravity for the cracked cross-section. S is the static moment of the cracked cross-section around the neutral axis and $\bar{A}$ is the area of the cracked cross-section. Thermal forces are described by the axial strain $\delta_T$ and the warping $\phi_T$. The center of gravity of the cracked cross-section lies on the distance of $\bar{x}$ from the compressed surface. External forces are expressed by the normal force N, the distance of which from the compressed surface is e'. The positive direction of e' is opposite to the direction of a.

The equation has a fundamental importance in Gurfinkel's method. Since all the cross-sectional quantities are functions of the distance of the neutral axis, a, the equation (15) is rather complicated to solve. We have done it by first determining a zone within which the solution lies. Then the exact position of the neutral axis is calcutated by applying Newton's iteration formula.

As mentioned earlier the central phases of the design procedure are to calculate design forces and then to use these forces to determine the reinforcement which fulfils the requirements of the limit state of cracking. It should be emphasized that the equation (15) is only used in the presented form after the first FEM-analysis, when the reduction of stiffness is considerable. Later the thermal forces are added to the other forces, and they only influence the value of N in the equation (15).

## 4. RE-EVALUATION OF STIFFNESS MATRICES

Basically, the coupling of structural analysis and design means re-evaluation of the stiffness of the structure and its influence on the response of the structure. In the beginning the structure is assessed to obey the theory of linear elasticity. However, after the first clearly anisotropic phenomenon, that is cracking, this assumption is not a real one. In our approach the anisotropy to the system is induced by the cracking and the orthotropic reinforcement. During the design procedure the stiffnesses are first presented in the coordinate system defined by the direction of cracking in an element. When the stiffnesses are updated, they are rotated back to the coordinate system of an element and used in the next FEM-analysis.

As mentioned earlier, the direction of cracking is defined separately to the bottom and top surface of an element. This is done by solving the equation /8/

$$\rho_y\,(1+n\,\rho_x)\,\tan^4\theta + n_x\,\rho_y\,\tan^3\theta\; - n_y\,\rho_x\,\tan\theta\; - \rho_x\,(1+n\rho_y) = 0 \qquad (16)$$

where $\theta$ is the direction of cracking, $\rho$ is the ratio of the steel area of the surface to the thickness of the cross-section, $n_x$ is the ration $N_x/N_{xy}$, and n is the ratio between Young's moduli of steel and concrete. For an element the direction of cracking is the combination of the directions obtained for the bottom and top surface.

The bending stiffness of the uncracked concrete is

$$K_{cB} = \frac{E_c t^3}{12(1-\nu_c^2)} \begin{bmatrix} 1 & \nu_c & 0 \\ & 1 & 0 \\ \text{symm.} & & (1-\nu_c)/2 \end{bmatrix} \qquad (17)$$

and in the directions of the reinforcing bars (x and y) the bending stiffness of the reinforcement is /15/

$$K_{sB}^{xy} = \frac{E_c n}{(1-\nu_c^2)} \begin{bmatrix} I_{sx} & \nu_c\sqrt{I_{sx}I_{sy}} & 0 \\ & I_{sy} & 0 \\ \text{symm.} & & (1-\nu_c)\sqrt{I_{sx}I_{sy}}/2 \end{bmatrix} \qquad (18)$$

In the equations $\nu_c$ is the Poisson's coefficient of concrete and $I_s$ is the moment of inertia of the reinforcement around the axis of center of gravity. The transformation matrix is

$$L = \begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\cos\theta\sin\theta \\ \sin^2\theta & \cos^2\theta & -2\cos\theta\sin\theta \\ -\cos\theta\sin\theta & \cos\theta\sin\theta & \cos^2\theta-\sin^2\theta \end{bmatrix} \qquad (19)$$

where $\theta$ is positive in the counter clockwise direction. In the new coordinate system 12, where the axis 1 is orthogonal to the direction of cracking, the bending stiffness matrix of the reinforcement may be written

$$K_{SB}^{12} = L K_{SB}^{xy} L^T \qquad (20)$$

It should be noted that in the new coordinate system the stiffness matrix has non-zero, off-diagonal terms. According to the equation (18) the off-diagonal terms and a diagonal term are presented as functions of the first two diagonal terms by determining four fictitious Poisson's coefficients. Similar transformations are also made for the membrane stiffness matrix.

After the first FEM-analysis the stiffness matrices of concrete are isotropic and need not be rotated. In the directions 1 and 2 new stiffnesses are now estimated by applying the cross-sectional method described earlier. Thus, only the first two diagonal terms are altered. The problem is how to modify the other terms.

It is clear that the membrane stiffness changes, if the center of gravity moves in the process. In that case it is assumed that the terms of the stiffness matrix can be expressed as functiona of the first two updated diagonal terms using the four fictitious Poisson's coefficients determined previously.

The coupling effect of concrete is determined so that only that part of concrete that is compressed in both directions is considered. The off-diagonal terms and the third diagonal term are calculated using the Poisson's coefficient of concrete. Hence, the deeper the cracks, the lesser the terms which are functions of Poisson's coefficient.

It should be emphasized that after the first updating of stiffnesses there are always four different stiffness matrices; the membrane and bending stiffness matrices for concrete and reinforcement. The membrane and bending stiffnesses required by the FEM-program are obtained as direct sums of the corresponding matrices of different materials.

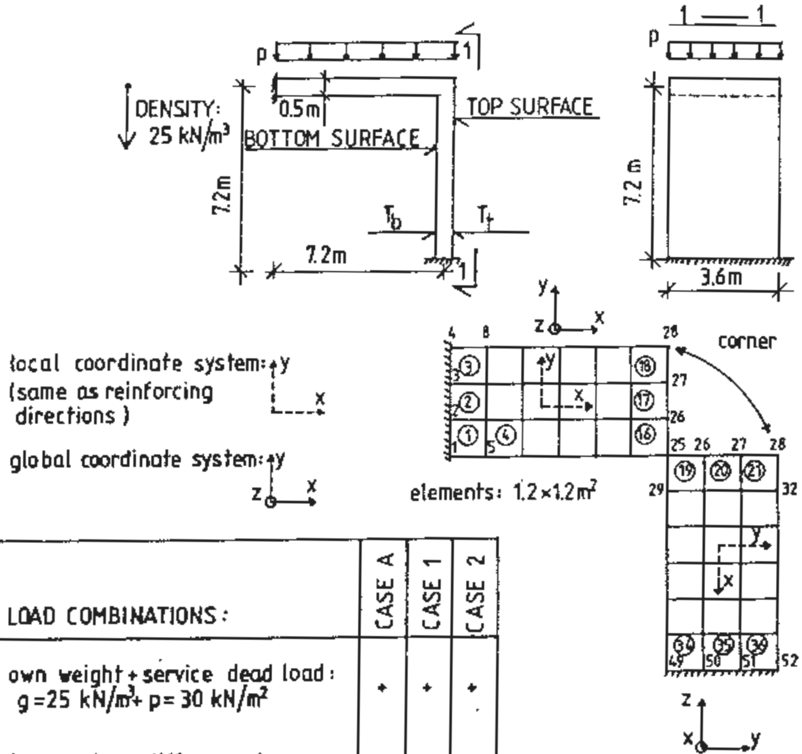
## 5. SAMPLE PROBLEM

In this chapter an application of the design procedure developed is presented. The structure analysed is shown in Figure 1. It consists of a vertical wall and a horizontal plate. The element model used and the load combinations examined are alao shown in Figure 1. The service dead load was 30 kN/m$^2$. It may act alone when we talk about Case A. In addition to the dead load there are two unsimultaneous temperature distributions, and so the load combinations Case A and Case B are obtained.

The following procedure of analysis has been used. The ultimate capacity of the structure is designed against the pure mechanical load, Case A. In the next the reinforcement obtained has been used as a minimum requirement. The reinforcement has a great effect on the stiffnesses of cracked reinforced concrete. Therefore the order in which the different load cases influence the structure might have a considerable effect on the final result. To avoid this difficulty, a design system is used in which the reinforcement required by the previous load case is used as a minimum requirement when analysing the next load case. In this connection it means that after Case A, Case 1 is examined and the reinforcement obtained is used as a minimum requirement during Case 2. The result of Case 2 is here called State B. Now the question arises, what influence Case 2 has on Case 1. Therefore the iteration is continued by using State B as a minimum requirement, and after the second analysis of Case 2, State C is received. If the difference between States B and C is small, the reinforcement that satisfies the design requirement is obtained. If not, the iteration should be continued.

The reinforcement required after State B and State C are presented in Figure 2. Figure 2 indicates a comparatively great increase of the reinforcement areas in the specified regions between the states. The increase in the total reinforcement amount is about 9 per cent. It might be interesting to know that if Cases 1 and 2 were dealt with separately and, after that, the determinative reinforcement areas from the separate cases were combined, the total reinforcement required would be underestimated by nearly 15 per cent. The worst consequence then would be that the fault was concentrated to certain areas.

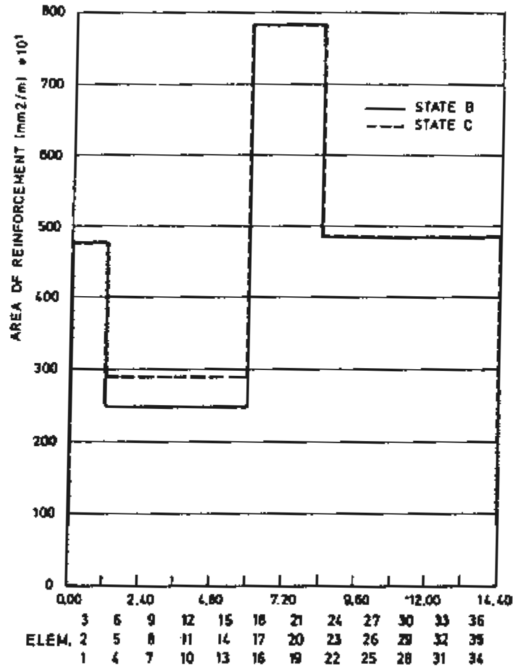
Figure 3 shows the characteristic crack widths that would appear, if the reinforcement had been in accordance with state B or C. As we can see, even the second iterative cycle is enough to balance the peaks of cracking to the limits allowed. We noticed that the greater the reinforcement amount is, the better the result converges. When limiting the crack widths, rather great reinforcement ratios occur. Thus, the final reinforcement can be obtained by two iterative cycles. If the structure is near the one-way system, two iteration cycles might not be enough. In this example the total amount of reinforcement required is approximately twice as big in the x-direction as in the y-direction (Fig. 1). The temperature difference also caused curvature in the y-direction.




| LOAD COMBINATIONS: | CASE A | CASE 1 | CASE 2 |
|---|---|---|---|
| own weight + service dead load: $g = 25\ kN/m^3 + p = 30\ kN/m^2$ | + | + | + |
| temperature difference 1: $T_b = -10°C$, $T_t = +25°C$ | | + | |
| temperature difference 2: $T_b = +25°C$, $T_t = -25°C$ | | | + |
| characteristic crack widths are limited: $w_k \leq 0.15mm$ | + | + | + |
| dimensioning in the ultimate limit state | + | | |

Figure 1. Geometry, the finite element model and load combinations of the problem analysed.

REINFORCEMENT IN THE LOCAL X-DIRECTION
IN THE TOP SURFACE




REINFORCEMENT IN THE LOCAL X-DIRECTION
IN THE BOTTOM SURFACE

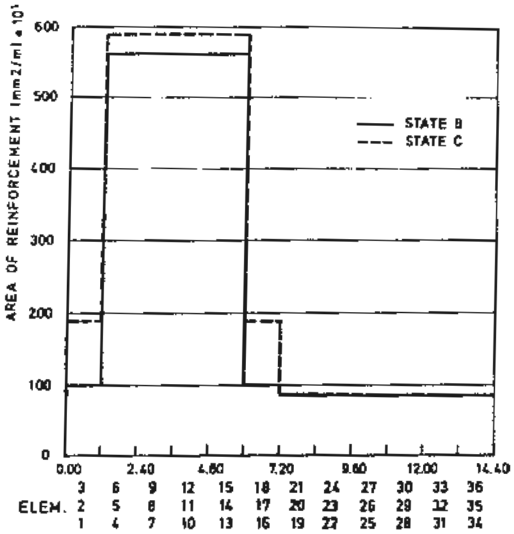



Figure 2. The reinforcement required after States B and C.

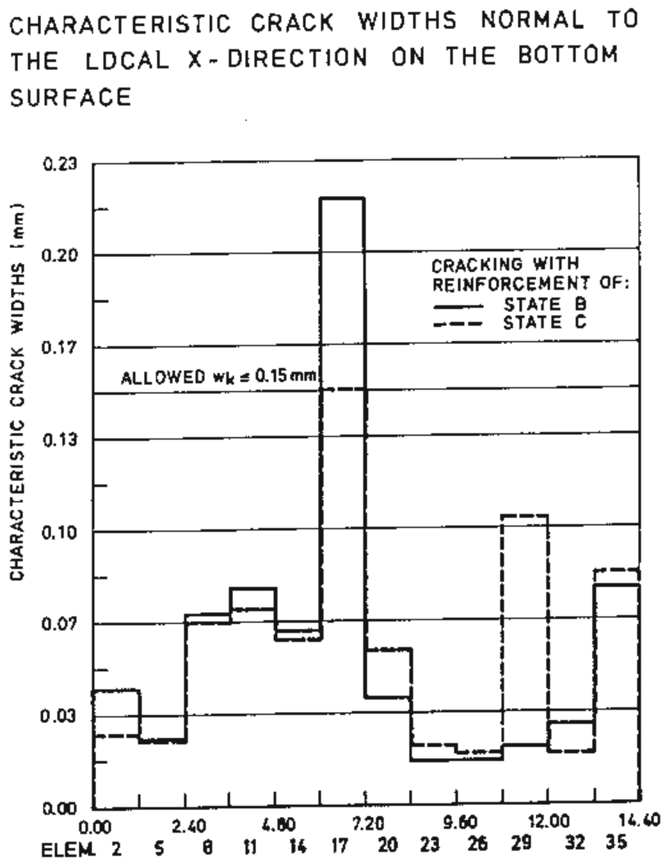



Figure 3. The characteristic crack widths, if the reinforcement is according to State B or C.

## 6. CONCLUSIONS

As a result of this study the IVODIM program has been made to analyse and design reinforced concrete shell structures. By the program the main non-linear effect of reinforced concrete, i.e. cracking, may be considered in an economic way in practical design. The method combines the present state-of-the-art of designing and analysing methods, and can be considered a first step to fully computerized designing, where the drafts are produced by CAD. The applications of the method have pointed out that, in the case of constraint load, especially the stiffness of cracked concrete must be considered before the optimal and economic reinforcement is obtained. As the sample problem confirmed the order of loading clearly influences the reinforcement in the limit state of cracking. It seems that if the order of loading is not considered, the present minimum reinforcement requirements turn out to be too small.

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