# LIMITATION OF CRACK WIDTHS BASED ON STOCHASTIC NATURE OF CRACK SPACING AND BONDING PROPERTIES OF REINFORCEMENT

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ABSTRACT

Dimensioning reinforced concrete structures against restraint effects is nearly always limiting the crack widths at an acceptable level. Because of the special nature of these action effects and the assumptions in the code crack width calculations the crack width seems to be underestimated by using the code method. Here a method is presented which takes into account the actual stress load and the cracking phase in the structure under observation. The method is based on physically true bonding properties of reinforcement and on the stochastic nature of concrete tensile strength and crack spacing. With certain assumptions the same method can be used for dimensioning against pure restraint effect, load effect, or combined restraint and load effect.

Keywords: concrete, cracking, restraint effect, randomness

## 1 INTRODUCTION

Normally reinforced concrete structures are subjected to load effects and restraint effects. The restraint effects result from compatibility conditions of the structure and their value is depending only on the stiffness of the structure. In most cases restraint effects or internal stresses are origineted from the environmental conditions or settlement of the supports. Prestressing of a statically indeterminate structures also cause restraint effects. Difference should be made between stresses caused by restraint effects and internal stresses, which are result from the compatibility conditions of the cross-section. Both these stresses can affect in the same or different directions mutually and with stresses caused by load effects i.e. external loads. In most practical cases these extra stresses are, however, ignored in the design and dimensioning of reinforced concrete structures.

This neglect, sometimes even deliberate, has often led to premature and unexpected cracking of concrete structures and to other defects arising from cracking. In accordance with quality concepts in modern building the cracking is a quality defect which should be avoided as far as possible or at least limited to an acceptable level.

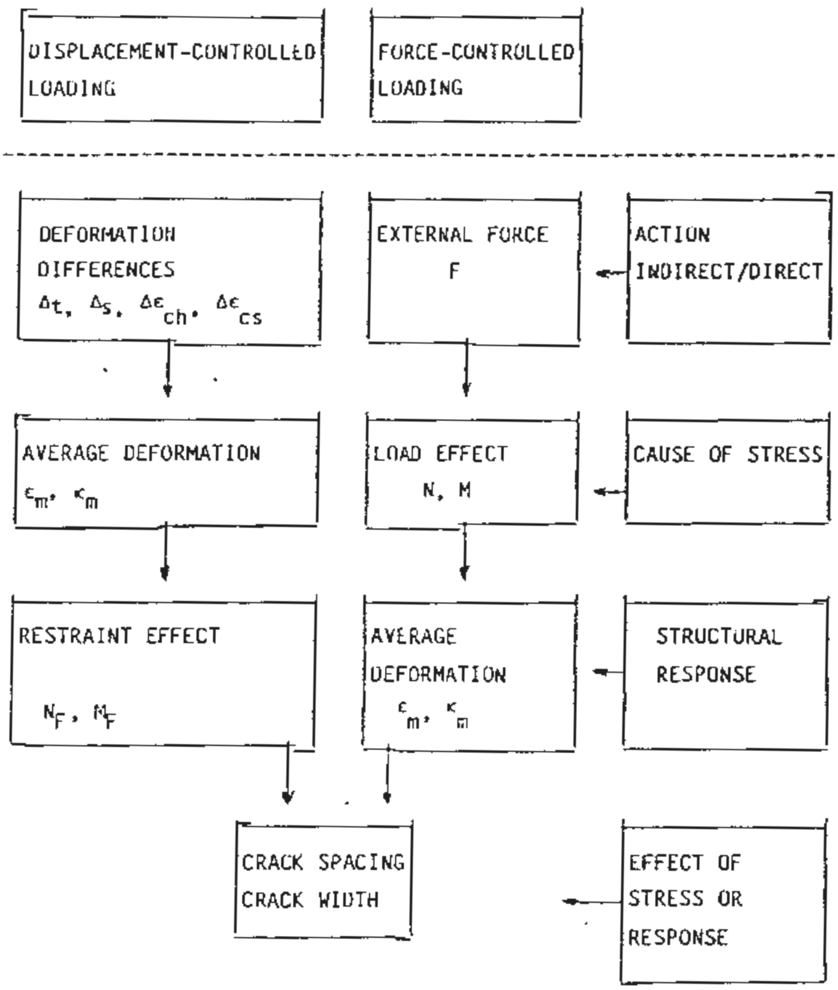


Fig 1. Contributing factors in different types of loading.

In this contribution a method, specially aimed for limitation of cracks caused by restraint effects, is presented. This method can also be used for limitation of cracks caused by load effects or combined load and restraint effects. The calculation of the actual values of the restraint and load effects is supposed to be known.

## 2 MODELS FOR CALCULATION

### 2.1 Deterministic models

Stress-strain relationship used in calculation is shown in Fig. 2.

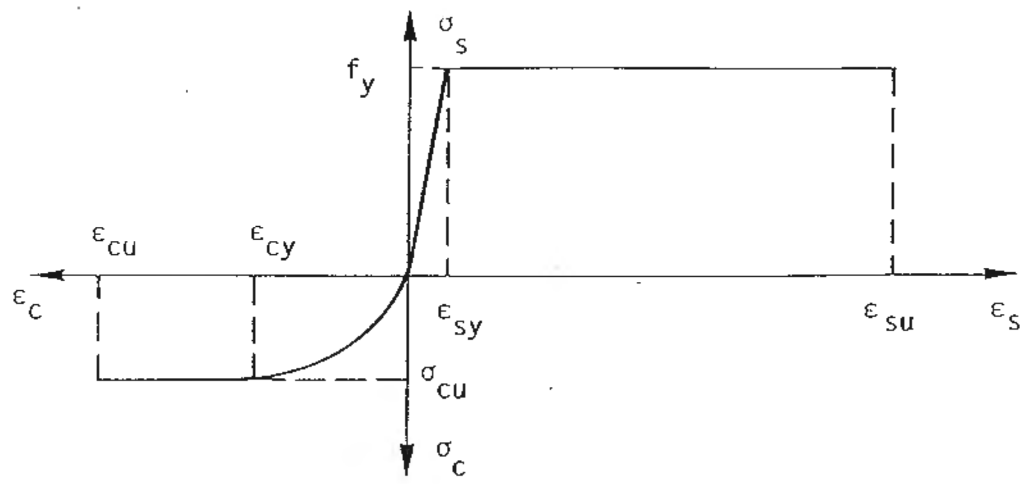


Fig. 2. $\sigma_c - \varepsilon_c$ and $\sigma_s - \varepsilon_s$ models.

For the tensile strength of concrete the following deterministic formula

$$f_{ct}(\eta) = \frac{0,06 + 4,52\ \eta}{0,1 + 6\ \eta} \cdot \frac{\alpha_i}{10} \sqrt[3]{(10K)^2} \quad (1)$$

was used. The formula includes the effects of stress gradient in first loading when the height of the beam is 400 mm. The suitable fraction can be determined by experimental factors $\alpha_i$, where i is a wanted fraction.

The interaction model between concrete and reinforcement is derived starting from the fundamental bond equations. By examining the elementary unit in Fig. 3 the following differential equation

$$\frac{d^2\delta}{dy^2} = \frac{\pi \cdot \Sigma\ \phi}{E_s\ A_s} \left(1 - \frac{n_s}{\eta_c \cdot A_{ceff}}\right) \tau_{cy} \quad (2)$$

is obtained.

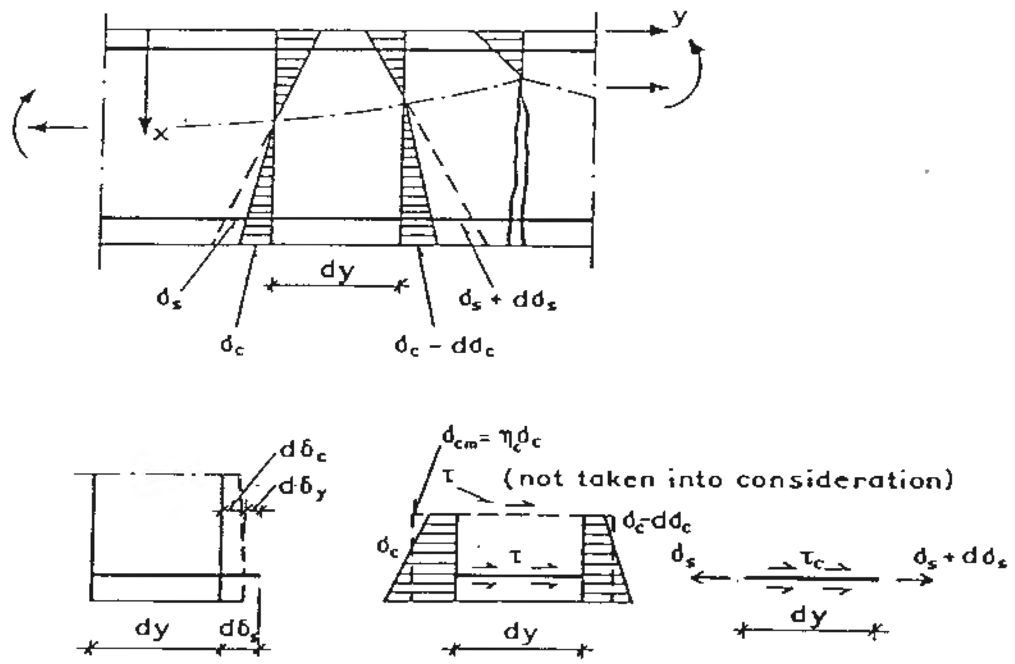


Fig. 3. Basic bond equations.

In order to find a solution for stresses and slip the relation $\tau_{cy} = f(\delta)$ between the local bond stress and slip should be known. Here the power function

$$\tau_y = \alpha K \cdot \delta^{1/\beta} \qquad (3)$$

is used. The parameters $\alpha$ and $\beta$ are determined experimentally for Finnish reinforcing steel bars. The values of $\alpha$ and $\beta$ are given in /2/.

Equation (2) is based on sliding bond. Since it is true that there are also components of non-sliding bond in the interaction mechanism, it is assumed here that there is a region by total length $a_0$ to both sides from the crack where no bond exists. This means from the point of view of cracking that the steel bar can elongate freely within this distance.

## 2.2 Stochastic models

The tensile strength of concrete is very essential in this connection because it determines the largest values of restraint effects.

In spite of the apparent homogeneity of concrete its tensile strength is distributed stochastically over a structure. In consequence of this random nature the formation of cracks also occurs in random places in those areas where the smallest local value of the tensile strength has been exceeded.

The nearer to the beginning the initiation of crack formation, the greater is the proportion of stochastic cracking. In this case the so-called bond-free zones between the cracks are at their largest, whereas near the stabilized cracking phase the bond-free zones, where new cracks are to be produced, are at their smallest. Expressed in statistical mathematics the stochastic nature of cracking means that the tensile strength of concrete has a certain standard deviation. If there were no deviation, i.e. if the tensile strength were constant, all cracks would be produced at the same moment at which the tensile strength is exceeded.

The determination of the location of a crack is impossible beforehand, but it is necessary that the random nature of the tensile strength of concrete and thereby the stochastic situation of cracks are known even by a very simple way, for example, in the calculations of crack widths and average deformations.

A clear picture of the random nature of tensile strength can be given by examining the tension zone of a reinforced concrete beam, for example. In principle, the tensile strength can be unequal in size in each cross-section of the tension zone, thus including all the values of the continuous density function of the hypothetical statistical distribution of the tensile strength. In practice, however, the density function is not continuous, since the tensile strength is regionally almost equally high and the regions in question are always limited in number. Thus the tension zone of a reinforced concrete bar or a beam can be assumed to consist of small pieces in the area of which the tensile strength is constant. If there were no reinforcing steel bars, the beam would fail after the first crack is produced or in a place where the tensile strength of concrete is locally at its lowest. Since the tensile force carried by the concrete is transmitted, however, to the reinforcement, the whole crack propagation corresponding to the tensile strength region can be described by increasing the stress uniformly over the whole area of the structure.

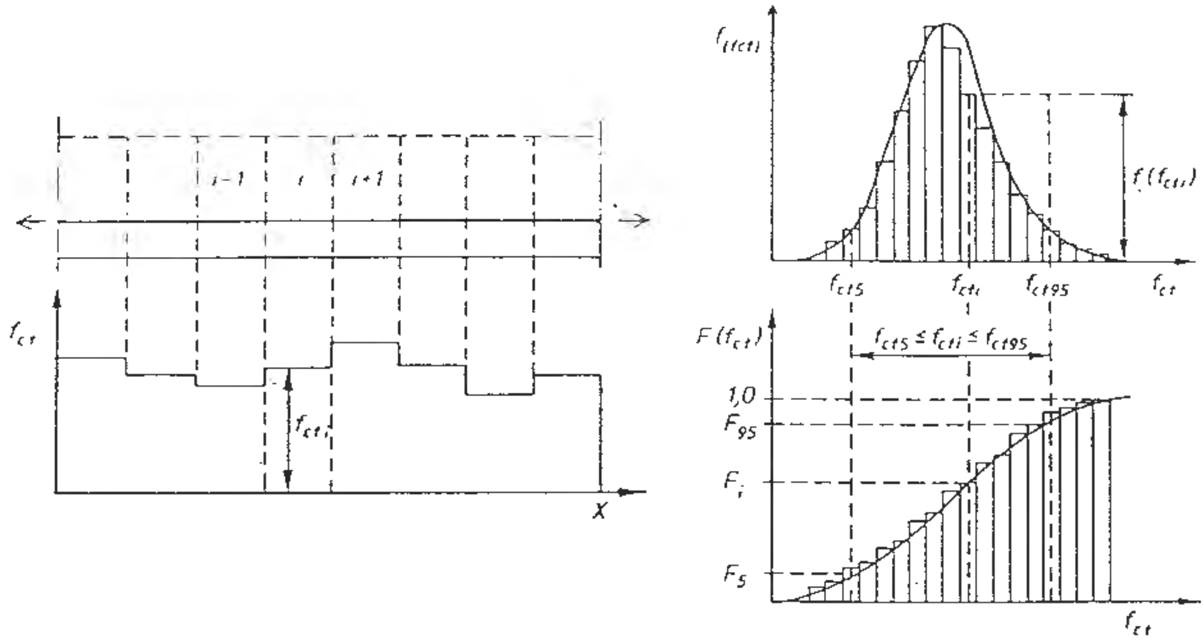


Fig. 4. Idealized tensile strength variations of concrete in the tension zone of a beam.

Owing to the random nature of the tensile strength cracks develop gradually in any of the strength zones including the probability area; these zones lie between the tensile strength values corresponding to the first crack formation and the stabilized cracking phase.

A statistical model can be formed for the tensile strength values of concrete, distributed as shown in Figure 4, which corresponds as far as possible to the actual distribution of the values, and which can be treated mathematically while accepting certain factors of uncertainty.

Distributions of random variables are estimated in statistical mathematics using the functions of the derivatives of the continuous cumulative distribution and probability functions or, in other words, by meams of density functions. It is very commonly assumed that the strength values of materials are normally distributed. The same applies here to the tensile strengths. Depending on the purpose, it is true that by selecting distribution models which are clearly incompatible with the facts very satisfactory results can also be achieved.

The said strength zone of the tensile strength, where the development of a crack pattern occurs, can be expressed mathematically, if it is assumed, for example, that the steel stress $\sigma_{sr,1}$ corresponds to the tensile strength which corresponds to the first crack formation, and that the steel stress $f_{sy}$ corresponds to the tensile strength which corresponds to the last crack formation, i.e. stabilized situation and that the strength interval is obtained by linear interpolation.

In this research a method is employed in which the steel strain value $\varepsilon_{sr,1}$ corresponds to the concrete tensile strength which corresponds to the first crack formation and $\varepsilon_{sy}$ corresponds to the initial stage of the stabilized cracking phase. Moreover, it is assumed that the changing over from the formation phase of the first cracks to the crack propagation phase corresponds to the mean value of the strength interval.

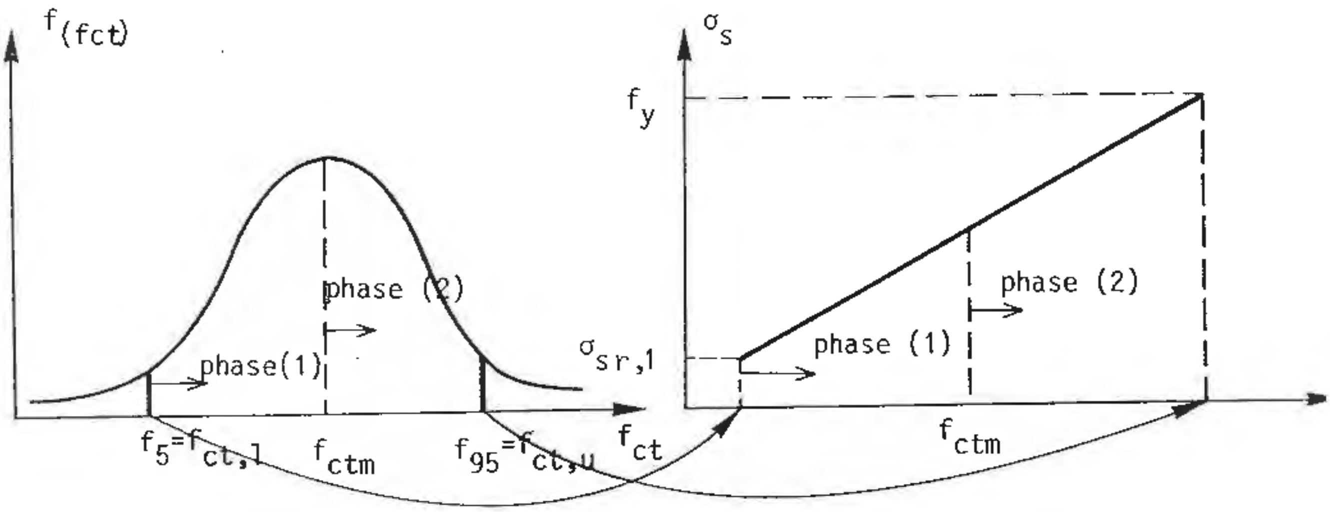


Fig. 5. Strength interval and corresponding steel stresses.

The density function corresponding to the strength zone determined in this way is limited at its lower end from the value $f_{ct,l}$ downwards and at its upper end from the value $f_{ct,u}$ upwards. This is mathematically arbitary but physically almost correct. This can be shown, for example, by examining test results.

Assuming the tensile strength of concrete to be constant is naturally not a physically realistic method of modelling the behaviour of a structure since tensile strength, as most material properties in different parts of the structure, has a stochastic nature with a certain dispersion. Owing to this the development of cracking can never be expressed as a precise mathematical formula. Instead of the use of probability calculation, statistical mathematics and certain assumptions concerning the nature of tensile strength must be relied upon. In other words, the development of cracking can be observed without assuming that the tensile strength of concrete is constant. By assuming that the tensile strength has a certain statistical distribution and estimates, the phase of crack development can be observed in a more realistic manner. This method of observation also leads to different deformation behaviour under force and displacement-controlled loads, even though the different types of behaviour also have points in common in this case.

Besides the length of the structure, the dispersion of the tensile stress of concrete also has an influence on differences in deformation behaviour caused by different types of stress.

The crack spacing is dependent on the degree of stress and on the random character of crack formation. It is also a random variable, as are all factors dependent on material properties and structural dimensions. Exact calculational determination of crack spacing, depending on the degree of stress, may be impossible. However, since the smallest and largest approximate numbers of cracks which stress can produce in a structure are known, the mean value of crack spacing in proportion to the minimum crack spacing corresponding to the desired probability can be calculated by making certain simplifications and statistical assumptions based on the properties of the tensile strength of concrete.

The number of cracks, and thus the average crack spacing, corresponding to a specific load in the first cracking phase and in the phase of crack formation is $s_r = \ell/n$. The greatest possible number of cracks that can be produced in the tension zone is m, which corresponds to the number of cracks in the stabilized cracking phase. Let us assume that tensile strength in each section has a constant value of $f_{cti}$ like in the tensile strength model. If the tension zone is now loaded with a stress of $\sigma_c = f_{cti}$ all sections in which $f_{ct} < f_{cti}$ will be cracked. The number of cracked sections can be calculated in terms of probability from the known cumulative distribution function $F(f_{ct})$ of tensile strength.

$$P(f_{ct} < f_{cti}) = F(f_{cti}) = \frac{\text{number of cracked sections}}{\text{total number of sections}}$$

or $$F(f_{cti}) = \frac{\text{number of cracks}}{\text{largest possible number of cracks}} = \frac{n}{m}$$

Where the crack spacing in the crack formation phase is dependent on stress, the variable $s_r = \ell/n$ is at its minimum in the stabilized cracking phase and the quantity $s_{rm} = \ell/m$ a constant. Thus $s_r/s_{rm} = m/n$ and when $\sigma_{ct} = f_{cti}$,

$$\frac{s_r}{s_{rm}} = \frac{1}{F(f_{cti})} \qquad (4)$$

The equation describes the development of cracks in the crack formation phase, taking into account the random nature of the tensile strength of concrete.

If we use the same principle of stress interval described in connection of concrete tensile strength model we can derive approximately

$$f_{cti} \approx f_{ct5} \frac{\varepsilon_{s2}}{\varepsilon_{sr,1}} \qquad (5)$$

In formula (5) it is assumed that the first crack initiates when the tensile strength of concrete corresponds to the value of 5 % fraction.

In Fig. 6 the relative crack spacings are shown. Calculations are made under assumption that the tensile strength of concrete is normally or uniformly distributed.

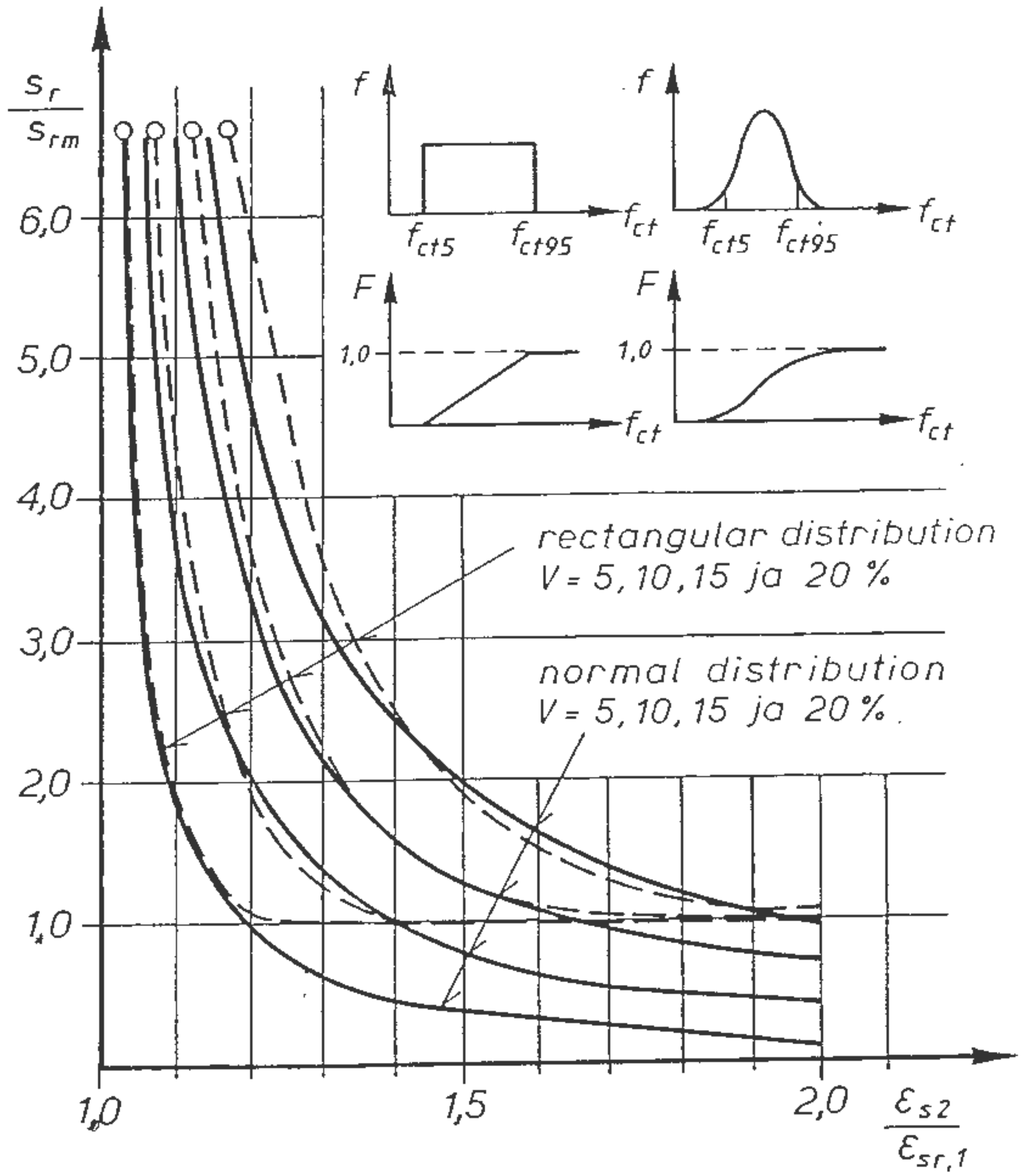


Fig. 6. Calculated crack spacings.

## 3 NUMERICAL SOLUTION

The general distributions and values of stresses and strains are obtained as a numerical solution to the differential equation derived from observations of basic bond equations of the elementary structural unit. Through these distributions and values and crack spacing dependent on the degree of stress determined on a probability basis the average value of strain in steel and the crack widths can be determined. The formation of new cracks is taken into account in the form of continuously diminishing crack spacing.

The deformations of the entire structure are calculated from the deformations of an elementary structural unit.

A section of beam, equivalent to the length of an average crack spacing corresponding to the degree of stress in question, comprises the elementary structural unit. It is therefore assumed that the average deformation in the region of the elementary unit is equal to the mean value of deformations of crack spacings of varying length.

Because of different boundary conditions in different cracking phases, the strain equations corresponding to each phase must be considered separately.

Calculating the crack widths is based on the fact that the difference between relative strains in the reinforcement and the concrete is integrated over the crack spacing in each state of stress. The crack spacing is thus a quantity that changes according to stress. Its length is determined by the interaction of concrete and reinforcement.

The interaction is based partly on the sliding bond and partly on the non-sliding bond, which causes a disturbance in the interaction near the cracks. Crack widths in different phases of cracking are determined as follows:

Initial cracking phase

$$w_m = 2 \cdot \int_{y=0}^{y=l_b} (\varepsilon_{sy} - \varepsilon_{cy})\, dy + 2 \int_{l_b}^{l_b+a_{0/2}} (\varepsilon_{sy} - \varepsilon_{sm})\, dy \qquad (6)$$

Crack propagation phase

$$w_m = 2 \int_{y=0}^{s_r/2} (\varepsilon_{sy} - \varepsilon_{cy})\, dy + 2 \int_{s_r}^{s_r+a_{0/2}} (\varepsilon_{sy} - \varepsilon_{cy})\, dy \qquad (7)$$

Stabilized cracking phase

$$w_m = s_{rm} \cdot \varepsilon_{sm} + a_0\ (\varepsilon_{s2} - \varepsilon_{sm}) \qquad (8)$$

where

$s_{rm} \approx l_b + 2\phi$ and
$a_0 = 4\phi$

4 DESIGN CURVES FOR CRACK LIMITATION

By taking into consideration the differences in action effect-deformation dependence, the design in both force-controlled and strain-controlled states of stress can be carried out on the same grounds as in the case where the moment is examined just before the initiation of a crack, at which moment both types of action effect-deformation curves join. It thus means that in the case of short beams or beams of finite length the jump in strain or action effect pertaining to the formation of each crack is not determined; on the contrary all the beams are supposed to be of finite length.

If the effect of each crack were considered separately, the task would be very difficult because the tensile strength of concrete is of a stochastic nature. The value of the restraint effect is then also on the safer side than immediately after crack initiation. Cracking due to restraint effects is thus controllable on the same grounds as cracking due to load effects. From a practical point of view it is suitable, however, to present the stresses caused by the load effects and the combined load and restraint effects as a function of steel stress at the crack, whereas when a mere restraint effect is acting a better parameter is an average strain.

Fig. 7 shows crack dimensioning curves for a mere restraint effect when assuming the value for the tensile strength of concrete to be divergent from zero. The diagrams are based on the tensile strength values presented in Chapter 2 and the values given in the same Chapter as a basis for the interaction between the concrete and the reinforcement. The crack width corresponding to constrained deformation known to be of a certain size can be determined for certain value couples of the bar diameter and the effective reinforcement area.

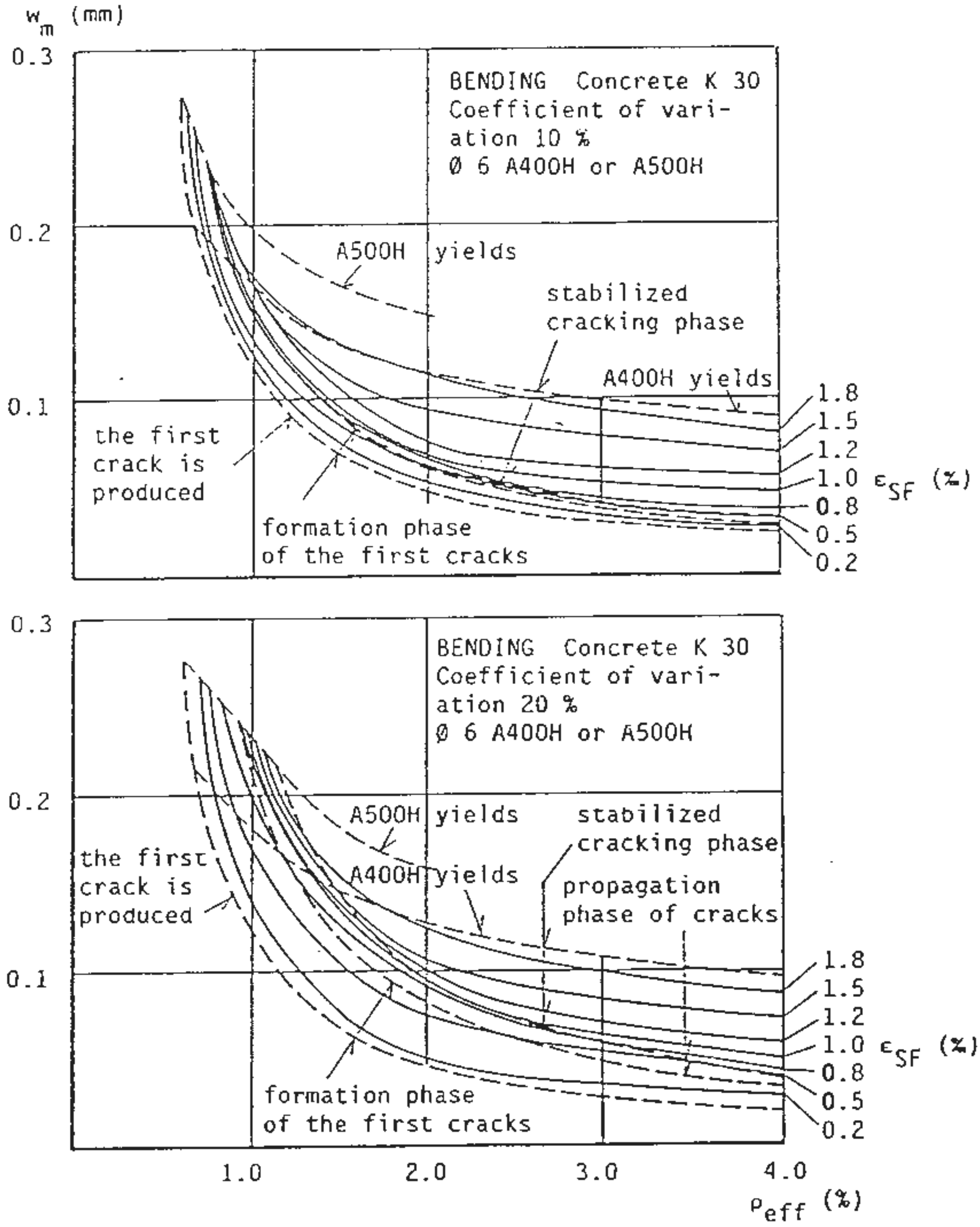


Fig. 7. Design diagram for crack width caused by restraint bending.

Before the first cracks are produced, the restraint and load actions can be treated separately and their effects can be combined or separated provided that there has not been time for creep and relaxation to occur. Subsequent to the formation of cracks neither the action effects nor their stresses can be distinguished from one another; on the contrary a resultant action effect is to be studied. In the cracked state the action sequence of the action effects is therefore of no great importance.

Design in the service state is also involved in most cases where the combined action effects are acting in the limitation of cracking to an acceptable level.

The diagram is based on the tensile strength values presented in Chapter 2 and on the bond parameters of the good bond zone as given in the same Chapter. The crack width in the weak bond zone can be 30 to 40 % larger. Further, the basis for the diagrams is the thickness of the concrete cover $c = 2\phi$, and the distance $a_0/2 = 2\phi$ on both sides of the crack as the disturbance length of the bond. The reinforcement level is presented in regard to the effective concrete area. If the strength of concrete essentially deviates from the values on which the formulae are based, it should be borne in mind that the crack width increases with increasing tensile strength but only when the compressive strength remains unchanged.

In Fig. 8 the curves corresponding to the initial cracking phase have been lengthened by a broken line up to the origin of the diagram. By means of these broken lines it is possible to determine the crack widths in some important cases that occur in practice, where the tensile strength can be assumed as zero.

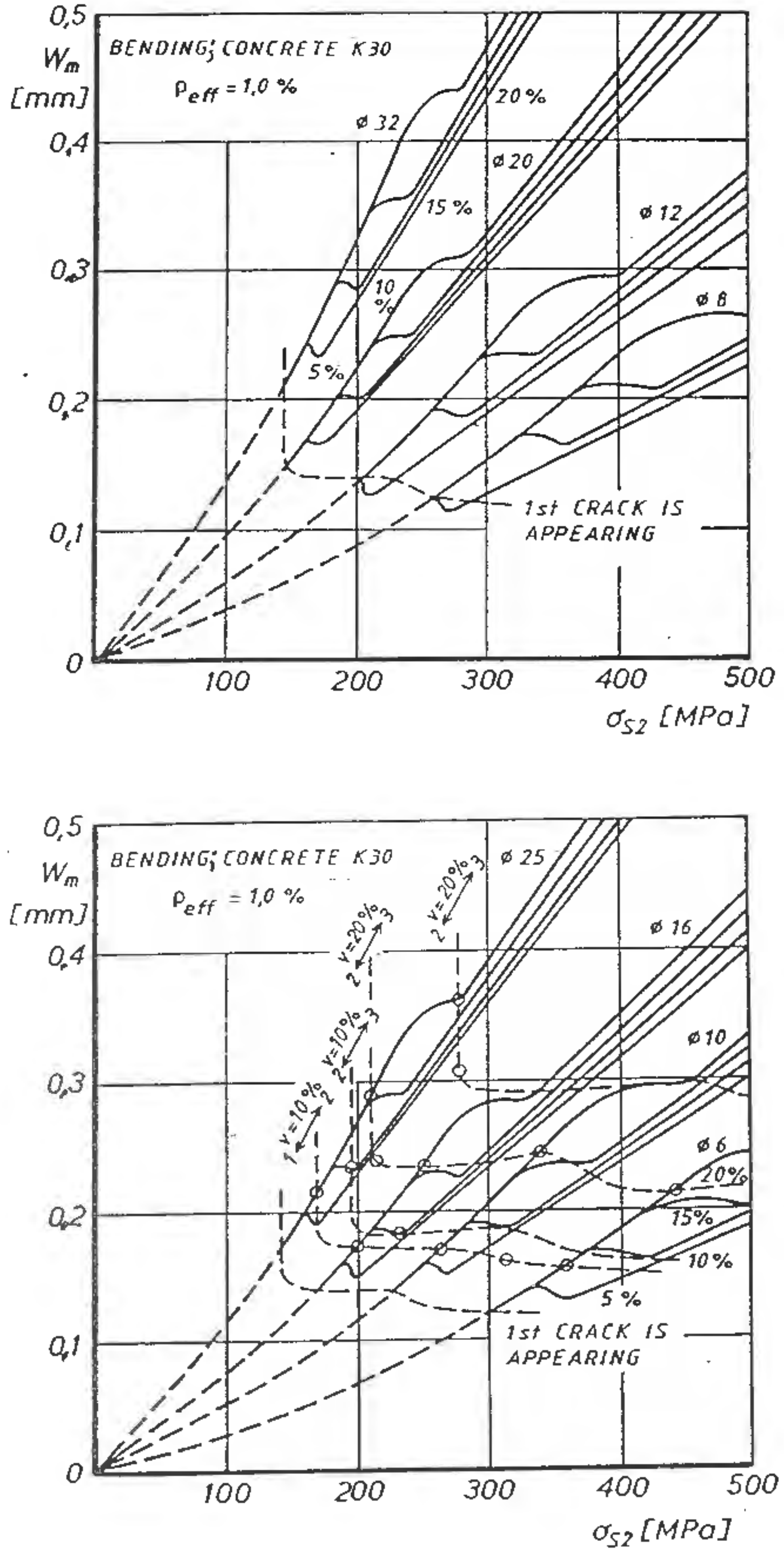


Fig. 8. Crack limitation diagram for bending caused by load effect or combined load and restraint effect.

## 5 CONCLUSIONS

1. It can be stated, according to calculations and tests, that the average steel strain cannot under normal working conditions attain the values required by the stabilized cracking phase. When larger amounts of steel are used the stabilized cracking phase can, in principle, be produced by external load, but not by usual temperature differences.

2. According to the method presented here, the boundary lies in the stress region where the crack width will be underestimated when using the formula based on the stabilized cracking phase.

3. In the initial cracking phase the crack width increases almost linearly as far as the steel stress $\sigma_{s2}$ is concerned. In the crack propagation phase, on the other hand, an increase in crack width is retarded or the widths in a certain stress zone do not increase at all, since there are only sliding bond zones between the cracks and the formation of new cracks takes place in such rapid succession that the average width of previous cracks has no time to increase. In the stabilized cracking phase the crack spacing is at its minimum and constant, and the average steel strain and thus crack width increase almost linearly; the lessening of the increase is not as sharp as in the initial cracking phase.

4. If the average steel stress or strain is used as a parameter for crack widths, the crack propagation phase is seen to be reduced by almost one curve. This is because the crack width in this phase does not increase much with an increasing average steel strain, particularly when the dispersion of the tensile strength of concrete is small.

## NOTATIONS

| | |
|---|---|
| $A_{ceff}$ | effective concrete cross-section |
| $A_s$ | steel area in cross-section |
| $E_s$ | moduls of elasticity of steel |
| $F$ | cumulative distribution function |
| $K$ | nominal strength of concrete |
| $M$ | moment of external load |
| $N$ | normal force of external load |
| $M_F$ | restraint moment |
| $N_F$ | restraint normal force |
| $P$ | probability |
| $a_0$ | detachment length ($4\phi$) |
| $f_{ct}$ | concrete tensile strength |
| $f_{ct,\ell}$ | concrete tensile strength corresponding to lower fractile |
| $f_{ct,u}$ | concrete tensile strength corresponding to upper fractile |
| $f_{ct,5}$ | concrete tensile strength corresponding to lower 5 % fractile |
| $l_b$ | anchorage length |
| $n$, $m$ | numbers |
| $n_s$ | ratio $E_s/E_c$ |
| $s_r$ | crack spacing |
| $s_{rm}$ | stabilized crack spacing |
| $\Delta t$ | temperature difference |
| $\Delta s$ | settlement of support |
| $w_m$ | mean crack width |
| $\alpha$, $\beta$ | parameters |
| $\delta$ | slip |
| $\varepsilon_c$ | strain in concrete |

| | |
|---|---|
| $\varepsilon_s$ | strain in steel |
| $\Delta\varepsilon_{ch}$ | creep difference |
| $\Delta\varepsilon_{cs}$ | shrinkage difference |
| $\varepsilon_{sr,1}$ | steel strain corresponding 1st crack |
| $\varepsilon_m$ | average strain |
| $\varepsilon_{s2}$ | steel strain at crack |
| $\varepsilon_{sF}$ | mean steel strain caused by restrained deformation |
| $\gamma_m$ | average curvature |
| $\eta$ | relative eccentricity |
| $\eta_c$ | parameter |
| $\sigma_c$ | stress in concrete |
| $\sigma_s$ | stress in steel |
| $\tau_{cy}$ | shear stress in concrete along y-axis |
| $\phi$ | bar diameter |

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